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LETTER TO THE EDITOR

On the spectrum of the Schrödinger operator for some many-particle systems

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Abstract. For the N-Coulomb particle Schrödinger operator with central charge Z there is a well known condition of stability $N<2Z+1$ obtained by Lieb. This estimate is extended to operators with slowly decreasing potentials.

In this letter we shall describe the lower bound of the spectrum of operators in $L_2(R^{3N})$ of the form

$$H_N=-\sum_{j=1}^{N}(\Delta_j+ZV(x_j))+\sum_{j<k}V(x_j-x_k) \qquad Z>0$$

i.e. H_N is the Hamiltonian of a non-relativistic quantum system (e.g. of an N-electron atom if the potentials are Coulombic). We shall suppose that

$$V(x)=b(|x|)/|x| \qquad x\in R^3 \qquad x\neq 0 \tag{1}$$

b being a positive, non-decreasing function such that $V(x)$ is monotonously decreasing when $|x|$ grows.

Let $E_N=E_N(Z)$ be the lowest eigenvalue of H_N. In the case of Coulomb potentials ($b=$ constant) it is well known (see [1]) that $E_N\geqslant E_{N+1}$ and there exists a critical number $N_{\max}$ such, that $E_N=E_{N_{\max}}$ where $N\geqslant N_{\max}$. From the physical point of view this means that the nucleus with charge Z cannot hold more than $N_{\max}$ electrons in the bound state (see [2–4]).

We shall extend this result to potentials of a more general type.

Theorem 1. Let $b''(r)\leqslant 0$. Then $N_{\max}<2Z+1$.

For $b=$ constant this result was obtained in [4]. We shall prove this theorem in a sequence of lemmas.

Lemma 1. Let $q(x)\in C^2(R^3)$, $q>0$. Then for any $f\in C_0^\infty(R^3)$ the equality

$$\int_{R^3}q(x)f(x)(-\Delta f(x))\,dx=\int_{R^3}q^{-1}|\nabla(qf)|^2\,dx+\int_{R^3}|f(x)|^2[\tfrac{1}{2}\Delta q-q^{-1}(\nabla q)^2]\,dx$$

holds. The proof follows from Green's formula.

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Corollary. Let $q\,\Delta q \geqslant 2(\nabla q)^2$. Then $(qf, -\Delta f) \geqslant 0$.

Lemma 2. Let the potential V be defined by (1) and $b''(r) \leqslant 0$. Then for any $f \in C_0^\infty(R^3)$ we have

$$(V^{-1}f, -\Delta f) \geqslant 0.$$

Proof. Let $q(x) = |x|/b(|x|)$. It is easy to check that

$$q\,\Delta q - 2(\nabla q)^2 = -|x|^2/b^{-3}b''(x) \geqslant 0.$$

Lemma 3. Let $d(t) = t/b(t)$. Then the inequality

$$d(\lambda + \mu) \leqslant d(\lambda) + d(\mu)$$

holds for every $\lambda > 0$ and $\mu > 0$. The proof is trivial.

Corollary. For all $x \in R^3$ and $y \in R^3$, then $V^{-1}(x+y) \leqslant V^{-1}(x) + V^{-1}(y)$.

Proof of theorem 1. We suppose that $E_N < E_{N-1}$ and denote by H_{N-1}^k the Hamiltonian of the system without particle x_k, i.e. for any fixed k $(l \leqslant k \leqslant N)$

$$H_N = H_{N-1}^k - \Delta_k - ZV(x_k) + \sum_{j \neq k} V(x_j - x_k).$$

Let $H_N f = E_N f$ and $(f, f) = 1$. Then

$$\begin{aligned} 0 &= (V^{-1}(x_k)f, (H_N - E_N)f) \\ &= (V^{-1}(x_k)f, (H_{N-1}^k - E_N)f) - (V^{-1}(x_k)f, \Delta_k f) \\ &\quad - Z + \left(V^{-1}(x_k)f, \sum_{j \neq k} V(x_j - x_k)f\right). \end{aligned}$$

Taking into account the inequalities $H_{N-1} \geqslant E_{N-1} > E_N$ we have, by applying lemma 2,

$$(V^{-1}(x_k)f, \sum V(x_j - x_k)f) < Z.$$

Hence

$$\sum_j \sum_{k \neq j} ((V^{-1}(x_k) + V^{-1}(x_j))\,V(x_k - x_j)f, f) < 2ZN$$

and by applying the corollary of lemma 3 we have

$$\sum_j \sum_{k \neq j} 1 < 2ZN.$$

Consequently $N(N-1) < 2ZN$ and $N - 1 < 2Z$. □

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